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ON THE CORRECTNESS OF THE BASIC PROBLEM OF DYNAMICS IN SYSTEMS WITH FRICTION*

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The paradoxical situation whereby the motion of a mechanical system with friction is not unique is studied on the basis of Chetayev's stability postulate, according to which the moiton must have a "particular kind of stability" /1/. As the disturbing factor we consider impacts which arise with the relative motion of rough surfaces and which lead to displacements, normal to the plane of frictional contact /2/. We demand of the true solution that it depend continuously on a small parameter, defining the size of these disturbances.

Painlevé's principle /3/ is usually employed to select the true solution from all possible solutions: "two rigid bodies, which under given conditions would not produce any pressure on one another, if they were ideally smooth, would likewise not act on one another if they were rought." This a priori principle has so far not obtained experimental confirmation, and its validity is uncertain. Moreover, the solution obtained using Painlevé's principle does not have the property of continuous dependence on the initial conditions /4/.

Our present approach leads to one way of resolving the situation whereby the basic problem of dynamics ceases to be correct in cases where the motion does not exist or is not unique in the given initial conditions: in either case we use the hypothesis that the impact is tangential /3, 5/. The resulting solution then differs from that obtained by Painlevé's principle.

Consider a mechanical system with configuration space $\mathbf{q} \in R^n \cap \{q_1 \geqslant 0\}$; the equation $q_1 = 0$ corresponds to frictional contact. When there are no impacts about the one-sided constraint $q_1 \geqslant 0$ the motion can be described by Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \mathbf{q}^{\cdot}} - \frac{\partial T}{\partial \mathbf{q}} = \mathbf{Q} + \mathbf{R}, \quad q_1\mathbf{R} = 0, \quad \mathbf{Q}, \mathbf{R} \in R^n \tag{1}$$

where T is the kinetic energy of the system, $\mathbf{Q} = \mathbf{Q}(\mathbf{q}, \mathbf{q}^{\cdot}, t)$ is the generalized force, and $\mathbf{R}$ is the reaction of the constraint. Solving (1) for the generalized accelerations, we obtain

$$\mathbf{q}^{\cdot\cdot} = \mathbf{A} + \mathbf{B}\mathbf{R}, \quad \mathbf{A} = \mathbf{A}(\mathbf{q}, \mathbf{q}^{\cdot}, t) \in \mathbf{R}^n, \quad \mathbf{B} = \mathbf{B}(\mathbf{q}, t) \in R^{n^2}, \quad B_{11} > 0 \tag{2}$$

where $\mathbf{A}$ and $\mathbf{B}$ will be assumed to be continuous functions of their arguments.

With $q_1 = q_1^{\cdot} = 0$ the reaction $\mathbf{R}$ satisfies the relations /6/

$$R_1 \geqslant 0, \; q_1^{\cdot\cdot} \geqslant 0, \; q_1^{\cdot\cdot}\mathbf{R} = 0 \tag{3}$$

the last of which expresses the passive nature of the reaction: its action cannot lead to weakening of the constraint. With impacts, the motion is described by the system /7/

$$\mathbf{q}^{\cdot}(t_0 + \Delta t) - \mathbf{q}^{\cdot}(t_0) = \mathbf{B}\mathbf{N}, \quad \mathbf{N} = \int_{t_0}^{t_0+\Delta t} \mathbf{R}(t)\, dt, \quad 0 \leqslant \Delta t \leqslant \tau \tag{4}$$

where t_0 is the time when impact starts, $\tau \ll 1$ is the duration of the impact, and $\mathbf{N}$ is the impact momentum.

We shall assume that the friction satisfies the Coulomb-Amonton law and that the connection between the reaction components is

$$R_j = f_j R_1, \; f_j = f_j(\mathbf{q}, \mathbf{q}^{\cdot}) \; (j = 2, \ldots, n) \tag{5}$$

where the dependence of f_j on $\mathbf{q}^{\cdot}$ is realized by means of the vector $\mathbf{v} = \mathbf{v}(\mathbf{q}, \mathbf{q}^{\cdot})$ of the relative velocity between the rigid bodies at their point of contact. On substituting (5)

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into (2) and into (4), we can write the equations of motion of systems with friction in the absence and in the presence of impact as

$$\mathbf{q}^{\cdot\cdot} = \mathbf{A} + R_1\mathbf{B}', \quad \mathbf{B}' = \mathbf{B}'(\mathbf{q}, \mathbf{q}^{\cdot}, t) \in \mathbf{R}^n \tag{6}$$

$$\mathbf{q}^{\cdot}(t_0 + \Delta t) - \mathbf{q}^{\cdot}(t_0) = \int_{t_0}^{t_0+\Delta t} R_1\mathbf{B}'(q, q^{\cdot}(t), t)\, dt \tag{7}$$

If there were no friction with $q_1 = 0$, $f_j \equiv 0$, then $B_1' = B_{11} > 0$ and, by (3), R_1 would be given by

$$R_1 = \max\ \{0, \ -A_1/B_1'\} \tag{8}$$

With $\Delta t = \tau$, (4) would take the form

$$\Delta q_1^{\cdot} = q_1^{\cdot}(t_0 + \tau) - q_1^{\cdot}(t_0) = B_1' N_1 = -(1 + \varkappa)\, q_1^{\cdot}(t_0) \tag{9}$$

where $\varkappa$ is the coefficient of restoration of the relative velocity on impact, $0 < \varkappa < 1$. Eq.(9) is attained for a value of the impact pulse $N_1 = -(1 + \varkappa)\, q_1^{\cdot}(t_0)\, (B_1')^{-1}$; since $B_1' > 0$ and $q_1^{\cdot}(t_0) < 0$, then $N_1 > 0$. Notice that $N_1 \to 0$ as $q_1^{\cdot}(t_0) \to -0$.

When friction is present, the coefficient B_1' can take zero or negative values, i.e., it may be impossible to find the reaction by means of (8), (9), and paradoxical situations arise in which the solution of the fundamental problem of dynamics does not exist or is not unique. For instance, if $B_1' \leqslant 0, A_1 < 0$, then Eq.(8) has no solution which satisfies conditions (3). In this case the hypothesis of tangential impact /3, 5/ is made; the motion is described in a small time interval by Eq.(7). As $\mathbf{q}^{\cdot}$ varies during the impact, B_1' becomes positive, with the result that indeterminacy of the motion is eliminated.

Notice that only the plane-parallel motion of rigid bodies was considered in /3, 5/, and the effect of tangential impact was invariably connected with stoppage of the relative slipping. In general, no such stoppage occurs, as can be shown by using, following /8/, the interpretation of impact by means of a curve in the $\xi = X/Z$, $\eta = Y/Z$, plane, where Z and X, Y are the normal and tangential components of the reaction. If the velocity of relative sliding $\mathbf{v}$ is non-zero, then the mapping point M moves over the circle $\xi^2 + \eta^2 = \mu^2$ (μ is the coefficient of sliding friction) in a direction which depends on the position of M relative to the hyperbola, at whose points $\mathbf{v}$ remains fixed. Corresponding to tangential impact we have a position of point M and of the origin O on different sides of the line $B_1' = 0$. During the impact the point M may pass into the domain $B_1' > 0$, without descending from the circle, if its trajectory does not then cut the hyperbola. The impact is then not accompanied by vanishing of the vector $\mathbf{v}$. Notice that, with tangential impact, $N_1 \not\to 0$ as $q_1^{\cdot}(t_0) \to -0$.

In the case when $A_1 = B_1' = 0$ any value of $R_1 \geqslant 0$ is admissible, while if $A_1 > 0, B_1' < 0$, condititions (3) are satisfied, not only by the value $R_1 = 0$ at which constraint $q_1 \geqslant 0$ is weakened, but also by one or even two values $R_1 > 0$, corresponding to cases of sliding or oscillation /4/. The above-mentioned Painleve's principle can be stated mathematically as

$$A_1 \geqslant 0 \Rightarrow R_1 = 0 \tag{10}$$

According to (10), the motion for which contact between the bodies ceases is regarded as true.

Let us study the influence on the motion with $q_1 = 0$ of collisions which are due to points of the body surfaces coming into contact with an initial velocity $q_1^{\cdot}(t_0) = -\varepsilon < 0$ as a result of the micro-relief of the surfaces. Such a collision will be described by (7). Denote by $\mathbf{q}(t, \varepsilon)$ the solution of system (1) with initial conditions $\mathbf{q}(t_0, \varepsilon) = \mathbf{q}^{\circ}, \mathbf{q}^{\cdot}(t_0, \varepsilon) = (-\varepsilon, q_2^{\cdot\circ}, \ldots, q_n^{\cdot\circ})$ and consider the behaviour of this solution as $\varepsilon \to +0$.

In the case $B_1'(C) > 0$, where $C \in R^{2n+1}$, $C = (\mathbf{q}^{\circ}, \mathbf{q}^{\cdot}(t_0, \varepsilon), t_0)$, the quantity N_1 vanishes in accordance with (9) along with ε and $\| C_\varepsilon - C \| = O(\varepsilon)$, where $C_\varepsilon = (\mathbf{q}^{\circ}, \mathbf{q}^{\cdot}(t_0, \varepsilon) + \Delta\mathbf{q}^{\cdot}, t_0)$, $\Delta\mathbf{q}^{\cdot}$ is the change of the velocity vector on impact at the instant t_0. Hence C is the same as $C_0 = \lim_{\varepsilon \to 0} C_\varepsilon$.

In the present case system (1) has a unique solution with $\varepsilon = 0$, since the reaction is uniquely defined by (8), (5). With $A_1(C) \geqslant 0$, for values of t sufficiently close to t_0, in (1), $\mathbf{R} = 0$ for $\mathbf{q}(t, \varepsilon)$ both for $\varepsilon = 0$ and for sufficiently small $\varepsilon > 0$ (the constraint weakens, $q_1 > 0$). Hence, for $t > t_0$, we have the estimate

$$\sum_{i=1}^{n} [|q_i(t, \varepsilon) - q_i(t, 0)| + |q_i^{\cdot}(t, \varepsilon) - q_i^{\cdot}(t, 0) - \Delta q_i^{\cdot}|] = O(\varepsilon) \tag{11}$$

In (11), $\mathbf{q}^{\cdot}$ is taken equal to the half-sum of its one-sided limits at points where it is discontinuous.

If $A_1(C) < 0$, we have $q_1 \equiv 0$ for the trajectory $\mathbf{q}(t, 0)$, while the trajectory $\mathbf{q}(t, \varepsilon)$ is of a vibratory-impact type. We construct a neighbourhood D_δ of the point C in such a way that, for $C' \in D_\delta$, we have

$$t_0 < t < t_0 + \delta,\ \| A(C') - A(C) \| < \delta,\ \| B'(C') - B'(C) \| < \delta \tag{12}$$

At points of the integral curve passing through C_ε, inside domain D_δ for sufficiently small $\delta > 0$ we have

$$q_1^{\cdot} = O(\varepsilon), \quad q_1 = O(\varepsilon^2) \tag{13}$$

For, in the interval between impacts $|q_1^{\cdot\cdot} - A_1(C)| < \delta$, so that, during this interval, the velocity modulus $|q_1^{\cdot}|$ increases by a factor not exceeding $[(A_1(C) - \delta)/(A_1(C) + \delta)]^{1/2}$, while on impact it is multiplied by an amount $\varkappa < 1$, whence follows the first of Eqs.(13); $\mathbf{q}_1$ then has order $\mathbf{q}_1^{\cdot 2}$.

We integrate (6) along the trajectory $\mathbf{q}(t, \varepsilon)$ in the interval (t_0, t^*), $t^* \leqslant t_0 + \delta$:

$$\mathbf{q}^{\cdot}(t^*) - \mathbf{q}^{\cdot}(t_0) = \int_{t_0}^{t^*} (\mathbf{A} + R_1\mathbf{B}')\,dt = \int_{t_0}^{t^*} \left(\mathbf{A} - \frac{A_1}{B_1'}\mathbf{B}'\right) dt + \int_{t_0}^{t^*} \frac{\mathbf{B}'}{B_1'} q_1^{\cdot\cdot}\,dt = I_1 + I_2 \tag{14}$$

The term I_1 in (14) has the same form for $\varepsilon = 0$ and $\varepsilon > 0$; for $\varepsilon = 0$, I_2 vanishes, while for $\varepsilon > 0$ it is a discontinuous function of t^*, which, by (13), has the estimate

$$\| I_2 \| \leqslant \varphi_\varepsilon(t^*), \quad \varphi_\varepsilon \in C_1[t_0, t_0 + \delta], \quad \varphi_\varepsilon(t_0) = 0, \quad \varphi_\varepsilon' \geqslant 0, \quad \varphi_\varepsilon(t_0 + \delta) = O(\varepsilon). \tag{15}$$

Since the solution of the equation

$$u(t) = \int_{t_0}^{t} \alpha u(s)\,ds + \varphi_\varepsilon(t), \quad u(0) = 0$$

has the form

$$u(t) = e^{\alpha t} \int_{t_0}^{t} e^{-\alpha s} \varphi_\varepsilon'(s)\,ds = O(\varepsilon)$$

then, following /9/, we can show that inequality (11) holds in the domain D_δ, where $\Delta q^{\cdot} = 0$.

We will now consider the singular case $B_1'(C) < 0$. By (7), collision with initial condition $q_1^{\cdot}(t_0) = -\varepsilon < 0$ can end only at a point C_ε such that $B_1'(C_\varepsilon) > 0$. Hence, apart from the dependence on $A_1(C)$, the impact is accompanied by a variation of $\mathbf{q}^{\cdot}$ which is finite as $\varepsilon \to +0$, i.e., is tangential. The limiting motion as $\varepsilon \to +0$ is also accompanied by a jump in the phase trajectory from point C to point C_0. Consequently, if we start from the condition, when choosing the true motion of the system, that it be continuous with respect to the parameter ε, then the motion is accompanied by tangential impact, apart from the dependence on $A_1(C)$. The situations whereby the solution of the basic problem of dynamics does not exist $(A_1(C) < 0)$ and is not unqiue $(A_1(C) \geqslant 0)$ are then identified, since the system behaviour after impact is defined by $A_1(C_0) \neq A_1(C)$.

At the end of the impact $B_1'(C_0) > 0$ and the motion has the same type as in the non-singular case, and in particular, (11) is satisfied, where $\Delta\mathbf{q}^{\cdot} \neq 0$. Notice that, by what has been said, this motion cannot be the same as one of the possible motions, constructed from values $B_1'(C)$, $A_1(C)$.

Finally, the case $B_1'(C) = 0$ in the above geometric interpretation of impact corresponds in the case when $v \neq 0$ to the initial position of the point M at the intersection of the straight line $B_1' = 0$ with the circle of friction. Depending on the disposition of the hyperbola, the point M is displaced either into the domain $B_1' > 0$, and then $C_0 = C$, or else into the domain $B_1' < 0$, and $C_0 \neq C$. Each of these cases is considered above.

Our results can be combined into the following theorem.

Theorem. Eq.(1), describing the motion of a system of rigid bodies in the presence of friction obeying the Coulomb-Amonton law, has a unique solution $\mathbf{q}(t, 0) \in C_1(t_0, t_1)$, which satisfies the initial conditions $\mathbf{q}(t_0, 0) = \mathbf{q}^{\circ}$, $\mathbf{q}^{\cdot}(t_0, 0) = \mathbf{q}^{\cdot\circ}$, $q_1^{\circ} = q_1^{\cdot\circ} = 0$, and has the property of continuous dependence in the sense of (11) on the small parameter ε, characterizing the collisions arising with friction. In the case when $B_1' \leqslant 0$, the generalized velocity $q^{\cdot}(t_0, 0)$ with $t = t_0$ can have a jump-type discontinuity, which is connected with an effect of tangential impact.

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INTEGRABLE CASES OF THE HAMILTON-JACOBI EQUATIONS AND DYNAMIC SYSTEMS REDUCIBLE TO CANONICAL FORM*

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Novel integrable cases of the Hamilton-Jacobi (HJ) equations are obtained. A method of reducing a class of non-autonomous dynamic systems to canonical form is given, and cases of their integrability are indicated. Comparison theorems are presented enabling the integrability of a dynamic system to be determined by observing the form of its Hamiltonian. The case of two bodies of variable mass in a resisting and gravitating medium are studied as an example.

1. The integration of canonical equations of motion is reduced to finding the complete integral of the corresponding HJ equation. The most interesting cases from the point of view of practical applications are the cases of integrability of the HJ, Liouville and Stäckel equations /1/ and their generalizations /2/. We shall establish new cases of integrability of the HJ equation of the form

$$p+\frac{1}{2}\sum_{i=1}^{n}g^{ii}p_i^2+\sum_{i=1}^{n}h^ip_i-U=0 \quad \left(p=\frac{\partial V}{\partial t},\quad p_i=\frac{\partial V}{\partial q_i}\right) \tag{1.1}$$

which generalize the result obtained by Yarov-Yarovoi /2/ and include the cases of integrability of Demin /3/, Liouville and Stäckel /1/.

Theorem 1.1. If the Hamiltonian is given by the formula

$$H=\frac{1}{2}\frac{\gamma}{b}\sum_{i=1}^{n}\frac{1}{a_i(q_i)}\left(p_i-\sum_{j=1}^{k}\varphi_j\frac{\partial\Phi_j}{\partial q_i}\right)^2-\sum_{j=1}^{k}\sigma_j\Phi_j-\frac{\gamma}{b}\sum_{i=1}^{n}U_i(q_i)+\Phi_0(t) \tag{1.2}$$

$$b=\sum_{i=1}^{n}b_i(q_i),\quad \Phi_j=\frac{1}{b}\sum_{i=1}^{n}\Phi_{ij}(q_i)\quad (j=1,2,\ldots,k) \tag{1.3}$$

$$\sigma_j=\varphi_j^{\cdot}-c_j\gamma\quad (j=1,2,\ldots,k\leqslant n) \tag{1.4}$$

where $a_i, b_i, U_i, \Phi_0, \Phi_{ij}$ are arbitrary continuous functions and $a_i\neq 0$, $b\neq 0$ and Φ_j are differentiable functions of the variables q_i, γ, σ_j, φ_j are continuous functions of time and c_j are arbitrary constants, then the HJ equation has a complete integral

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